Characterization of Cyclic and Separating Vectors and Application to an Inverse Problem in Modular Theory II. Semifinite Factors

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Abstract

This paper generalizes the results obtained in an earlier paper ([Bol]) for finite factors to infinite but still semifinite factors. First we give a characterization of cyclic and separating vectors for infinite semifinite factors in terms of operators associated with this vector and being affiliated with the factor. Further we show how this operator generates the modular objects of the given cyclic and separating vector generalizing an idea of Kadison and Ringrose. With the help of these results we can show that the second simple class of solutions for the inverse problem constructed in [Bol] never exists in infinite semifinite factors. Finally we give a classification of the solutions of the inverse problem in the case of modular operators having pure point spectrum completely analoguous to the finite case.

1 The Inverse Problem in Modular Theory

Let $\mathcal{M}_0$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}_0$ with a cyclic and separating vector u_0. Then modular theory shows the existence of a modular operator Δ_0 and a modular conjugation J_0 (the modular objects (Δ_0, J_0)) belonging to the vector u_0. In this paper we examine the inverse problem of constructing algebras $\mathcal{M}$ having the same cyclic and separating vector and modular objects as $\mathcal{M}_0$:

The Inverse Problem

Let (Δ_0, J_0) be the modular objects for the von Neumann algebra $\mathcal{M}_0$ with cyclic and separating vector u_0. Characterize all von Neumann algebras $\mathcal{M}$ isomorphic to $\mathcal{M}_0$ with the following properties:

1. u_0 is also cyclic and separating for $\mathcal{M}$,
2. (Δ_0, J_0) are the modular objects for $(\mathcal{M}, u_0)$.

Let $NF_{\mathcal{M}_0}(\Delta_0, \mathrm{J}_0, u_0)$ denote all solutions $\mathcal{M}$ of the inverse problem.

In [Bol] the following theorems were shown:

Theorem 1.1. *Let $(\mathcal{M}_0, \mathcal{H}_0)$ be a finite von Neumann factor. Let further $u \in \mathcal{H}_0$. Then there is exactly one operator $\mathrm{T}_u \eta \mathcal{M}_0$ associated with the vector u, s.t. $u = \mathrm{T}_u u_{\mathrm{tr}}$ where $u_{\mathrm{tr}} \in \mathcal{H}_0$ is a cyclic trace vector. This operator has the following properties:*

1. $\mathrm{tr}(\mathrm{T}_u \mathrm{T}_u^*) = \mathrm{tr}(\mathrm{T}_u^* \mathrm{T}_u) < \infty$.
2. *u is cyclic, iff T_u is injective.*
3. *u is separating, iff T_u has dense range.*
4. *u is cyclic and separating iff T_u is injective and has dense range, i.e. iff T_u is invertible.*

Theorem 1.2. *Let $\mathrm{T}\eta\mathcal{M}_0$. Then there is a vector $u \in \mathcal{H}_0$ s.t $\mathrm{T} = \mathrm{T}_u$ in the sense of Theorem 1.1 iff $\mathrm{tr}\,\mathrm{T}\mathrm{T}^* = \mathrm{tr}\,\mathrm{T}^*\mathrm{T} < \infty$.*

In the second section of this paper these theorems were generalized to infinite semifinite factors. For this purpose we first consider a special case of such factors, a matrix algebra of finite factors, where the trace vector is replaced by a sequence of vectors, constructed from the trace vectors of the constituting factors.

With the help of this result we show the analogue of the following result, also obtained in [Bol] for finite factors:

Theorem 1.3. *Let $\mathcal{M}_0$ be a finite von Neumann factor with cyclic and separating vector $u_0 \in \mathcal{M}_0$ and cyclic trace vector $u_{\mathrm{tr}} \in \mathcal{H}_0$. Let further $\mathrm{T}_{u_0}\eta\mathcal{M}_0$ be the invertible operator corresponding to u_0 and $\mathrm{T}_{u_0} = \mathrm{HV} = (\mathrm{T}_{u_0}\mathrm{T}_{u_0}^*)^{1/2}\mathrm{V}$ the polar decomposition of T_{u_0}. Then we can calculate the modular objects (Δ_0, J_0) of $(\mathcal{M}_0, u_0)$ as follows:*

$$\mathrm{J}_0 = \mathrm{J}\mathrm{V}^*\mathrm{J}\mathrm{V}\mathrm{J} = \mathrm{V}\mathrm{J}\mathrm{V}^*,$$

where J is the conjugation corresponding to u_{tr}, and

$$\Delta_0 = \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \mathrm{H}_0,$$

where $\mathrm{H}_0 = \mathrm{H}^2 = \mathrm{T}_{u_0}\mathrm{T}_{u_0}^*$.

Then we will be in exactly the same situation as in the finite case, and can examine the inverse problem as in [Bol]. In contrast to that case the second simple class of solutions will never exists in this case (s. §4), but the classification of the solutions in the pure point spectrum case will be the same.

Notice that in this paper all Hilbert spaces are separable, i.e. the von Neumann algebras are countably decomposable.

2 Characterization of Vectors by Affiliated Operators

In this section we consider infinite but still semifinite factors, i.e. $\mathcal{M}_0$ is of type I_∞ or II_∞. In this case we have no trace vectors left. But nevertheless we can make a similar construction as in the finite case by considering the infinite factor as an infinite matrix of finite factors and using the results presented in [Bol].

As a model for such a matrix of finite factors we examine now the semifinite factor $\mathcal{R} = \mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C}$ on $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty$, where $(\mathcal{T}, \mathcal{H})$ is a finite factor possessing a cyclic and separating vector and $\mathcal{H}_\infty$ is a infinite dimensional separable Hilbert space which we can identify with $l_2(\mathbb{N})$. Now $\mathcal{R}$ is an (infinite) type I (II) factor, if $\mathcal{T}$ is type I (II). Further, since $(\mathcal{T}, \mathcal{H})$ is finite and possesses a cyclic and separating vector, it possesses a cyclic trace vector $u_{\mathrm{tr}} \in \mathcal{H}$ (cf. [KR86, Th. 8.2.8, Lem. 7.2.8]).

In the following we consider the elements of $\mathcal{K}$ as infinite dimensional matrices $u = (u_i^k)_{i,k\in\mathbb{N}}$ with entries $u_i^k \in \mathcal{H}$ s.t. $\sum_{i,k} \left\| u_i^k \right\|^2 < \infty$, where the lower index corresponds to the second component of the tensor product and the upper to the third, resp. Then we can write the elements of $\mathcal{R}$ as matrices $\mathrm{T} = (\mathrm{T}_{li})_{l,i\in\mathbb{N}}$ with entries $\mathrm{T}_{li} \in \mathcal{T}$, where

$$\mathrm{T}u = (\sum_i \mathrm{T}_{li} u_i^k)_l^k.$$

Then the commutant $\mathcal{R}^{'}$ of $\mathcal{R}$ is $\mathcal{T}^{'} \otimes \mathbb{C} \otimes L(\mathcal{H}_\infty)$, where we can write an element in $\mathcal{R}^{'}$ as $\mathrm{T}^{'} = (\mathrm{T}^{'lk})_{l,k\in\mathbb{N}}$ with entries $\mathrm{T}^{'lk} \in \mathcal{T}^{'}$, where

$$\mathrm{T}^{'} u = (\sum_k \mathrm{T}^{'lk} u_i^k)_i^l.$$

For the proofs in this section the following subalgebras of $\mathcal{R}$ and $\mathcal{R}^{'}$ are important:

$$\mathcal{R}_0 := \{\mathrm{M} = (\mathrm{M}_{ij})_{ij} \in \mathcal{R} | \mathrm{M}_{ij} \neq 0 \text{ for only finitely many } i, j \in \mathbb{N}\}$$
$$\mathcal{R}_0^{'} := \{\mathrm{M}^{'} = (\mathrm{M}^{'ij})^{ij} \in \mathcal{R}^{'} | \mathrm{M}^{'ij} \neq 0 \text{ for only finitely many } i, j \in \mathbb{N}\}.$$

Now we define the following sequence of vectors in $\mathcal{K}$, which is the analogue to the trace vector:

$$v_k := (\delta_i^j \delta_{ik} u_{\mathrm{tr}})_i^j.$$

Further we define

$$D_0 := \mathrm{lin}\{\mathrm{M} v_k | \mathrm{M} \in \mathcal{R}, k \in \mathbb{N}\} \subset \mathcal{K}$$

and

$$D_0^{'} := \mathrm{lin}\{\mathrm{M}^{'} v_k | \mathrm{M}^{'} \in \mathcal{R}^{'}, k \in \mathbb{N}\} \subset \mathcal{K}.$$

Now the trace tr of $\mathcal{R}$, which is a n.s.f. tracial weight, is $\mathrm{tr} = \mathrm{tr}_{\mathcal{T}} \otimes \mathrm{tr}_{L(\mathcal{H}_\infty)}$, where $\mathrm{tr}_{\mathcal{T}}$ is the trace on $\mathcal{T}$ and $\mathrm{tr}_{L(\mathcal{H}_\infty)}$ the standard trace on $\mathcal{H}_\infty$. It can be written with the help of the vectors (v_k):

$$\begin{aligned}\mathrm{tr}(\mathrm{M}) &= \sum_k \mathrm{tr}_{\mathcal{T}}(\mathrm{M}_{kk}) = \sum_k \left\langle \mathrm{M}_{kk} u_{\mathrm{tr}} | u_{\mathrm{tr}} \right\rangle \\ &= \sum_k \left\langle \mathrm{M} v_k | v_k \right\rangle \\ &= \sum_k \int \lambda d \left\| \mathrm{E}^{\mathrm{M}}_\lambda v_k \right\|^2 \quad \forall \mathrm{M} = (\mathrm{M}_{ij}) \in \mathcal{R}^+,\end{aligned}$$

where $\mathrm{M} = \int \lambda d\mathrm{E}^{\mathrm{M}}_\lambda$ is the spetral measure of M. As in [Bol] we can continue the trace to all the positive closed operators A affiliated with $\mathcal{R}$ by

$$\mathrm{tr}(\mathrm{A}) := \sum_k \int \lambda d \left\| \mathrm{E}^{\mathrm{A}}_\lambda v_k \right\|^2, \tag{2.1}$$

where $\mathrm{E}^{\mathrm{A}}_\lambda$ is the spectral measure of A.

Now we can associate an operator $\mathrm{T}_{ij} \eta \mathcal{T}$ with every component u_i^j of a vector $u = (u_i^j)_i^j \in \mathcal{H}$, s.t. $u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{T_{ij}})$ and $\mathrm{T}_{ij} u_{\mathrm{tr}} = u_i^j$ (cf. [Bol]). These operators give rise to a linear operator $\tilde{\mathrm{T}}_u$ defined by

$$\begin{aligned}\tilde{\mathrm{T}}_u : \mathcal{D}(\tilde{\mathrm{T}}_{\mathrm{u}}) := D_0^{'} \subset \mathcal{K} \to \mathcal{K}& \\ \mathrm{M}^{'} v_k \mapsto \tilde{\mathrm{T}}_u \mathrm{M}^{'} v_k := (\mathrm{T}_{ik} {\mathrm{M}^{'}}^{jk} u_{\mathrm{tr}})_i^j&.\end{aligned} \tag{2.2}$$

Now we can prove

Lemma 2.1. *Let* $\tilde{\mathrm{T}}_u$ *defined by* (2.2). *Then* $\tilde{\mathrm{T}}_u$ *is densely defined and closable. Let* T_u *be its closure. Then* $D_0^{'} \subset \mathcal{D}(\mathrm{T}_{\mathrm{u}}^*)$, T_u *is affiliated with* $\mathcal{R}$, *and*

$$\sum_k \mathrm{T}_u v_k = u,$$

where the convergence is absolute.

Proof. 1. First we must show that $\tilde{\mathrm{T}}_u$ is well defined. Observe first that

$$\begin{aligned}\sum_{i,j} \left\| \mathrm{T}_{ik} \delta^{jk} u_{\mathrm{tr}} \right\|^2 &= \sum_i \left\| \mathrm{T}_{ik} u_{\mathrm{tr}} \right\|^2 \\ &= \sum_i \left\| u_i^k \right\|^2 < \infty,\end{aligned}$$

i.e. $\tilde{\mathrm{T}}_u v_k \in \mathcal{K}$ for every $k \in \mathbb{N}$. Let now $\mathrm{M}^{'} \in \mathcal{R}^{'}$ be arbitrary. Then

$\mathrm{M}'\tilde{\mathrm{T}}_u v_k \in \mathcal{K}$ and

$$\begin{aligned}
\infty > \left\|\mathrm{M}'\tilde{\mathrm{T}}_u v_k\right\|^2 \\
&= \sum_{i,j} \left\|\mathrm{M}'^{jk}\mathrm{T}_{ik}u_{\mathrm{tr}}\right\|^2 \\
&= \sum_{i,j} \left\|\mathrm{T}_{ik}\mathrm{M}'^{jk}u_{\mathrm{tr}}\right\|^2 \quad \text{(cf. [Bol, Prop.2.1.])} \\
&= \left\|\tilde{\mathrm{T}}_u\mathrm{M}' v_k\right\|^2
\end{aligned} \tag{2.3}$$

for every $k \in \mathbb{N}$, hence $\tilde{\mathrm{T}}_u$ is well defined.

2. Now we show that $\mathcal{D}(\tilde{\mathrm{T}}_{\mathrm{u}})$ is dense in $\mathcal{K}$. First the elements with only finitely many entries not 0 are dense in $\mathcal{K}$. Further every such element is a linear combination of elements of the type $(u_i^j\delta_{ik})_i^j$, again all but a finite number equal 0. Since $u_{\mathrm{tr}} \in \mathcal{H}$ is cyclic for $\mathcal{T}'$, we can approximate these elements by elements of the form $(\mathrm{M}'^{jk}\delta_{ik}u_{\mathrm{tr}})_i^j =: \mathrm{M}'v_k$ with $\mathrm{M}' = (\mathrm{M}'^{jk}\delta^{ik})^{ji} \in \mathcal{R}_0' \subset \mathcal{R}'$, hence $\mathcal{D}(\tilde{\mathrm{T}}_{\mathrm{u}}) = D_0'$ is dense in $\mathcal{K}$.

3. In this step we want to show that $\tilde{\mathrm{T}}_u$ is closable. Let $x = \mathrm{M}'v_k \in \mathcal{D}(\tilde{\mathrm{T}}_{\mathrm{u}})$, $y = \mathrm{N}'v_j \in \mathcal{D}(\mathrm{S}) := D_0'$ $(k, j \in \mathbb{N})$, where $\mathrm{S} := (\mathrm{T}^*_{li})_{i,l}$ is defined analogously to $\tilde{\mathrm{T}}_u$, hence it is a densely defined operator, too (All T^*_{li} are closed operators affiliated with $\mathcal{T}$ and $u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{T}^*_{\mathrm{li}})$, cf. [Bol, Prop. 2.1.], and $\|\mathrm{T}^*_{li}u_{\mathrm{tr}}\|^2 = \|\mathrm{T}_{li}u_{\mathrm{tr}}\|^2$).

   Now

$$\begin{aligned}
\langle\tilde{\mathrm{T}}_u x|y\rangle &= \langle\tilde{\mathrm{T}}_u\mathrm{M}'v_k|\mathrm{N}'v_j\rangle \\
&= \sum_{i,l}\langle\mathrm{T}_{ik}\mathrm{M}'^{lk}u_{\mathrm{tr}}|\mathrm{N}'^{lj}\delta_{ij}u_{\mathrm{tr}}\rangle \\
&= \sum_{l}\langle\mathrm{M}'^{lk}u_{\mathrm{tr}}|\mathrm{T}^*_{jk}\mathrm{N}'^{lj}u_{\mathrm{tr}}\rangle \\
&= \sum_{i,l}\langle\delta_{ik}\mathrm{M}'^{lk}u_{\mathrm{tr}}|\mathrm{T}^*_{ji}\mathrm{N}'^{lj}u_{\mathrm{tr}}\rangle \\
&= \langle x|\mathrm{S}y\rangle.
\end{aligned}$$

   This shows $y \in \mathcal{D}(\tilde{\mathrm{T}}^*_{\mathrm{u}})$, $\tilde{\mathrm{T}}^*_u y = \mathrm{S}y$, and $\mathrm{S} \subset (\tilde{\mathrm{T}}_u)^*$, hence $\tilde{\mathrm{T}}_u$ is closable. This shows also, that $D_0' \subset \mathcal{D}((\tilde{\mathrm{T}}_{\mathrm{u}})^*) = \mathcal{D}(\mathrm{T}^*_{\mathrm{u}})$.

4. To show that T_u is affiliated with $\mathcal{R}$, let $\mathrm{U}' = (\mathrm{U}'^{ij})_{i,j\in\mathbb{N}} \in \mathcal{R}'$ be a

unitary. Then $\mathrm{U}' D_0' = D_0'$. Let now $x = \mathrm{M}' v_k \in D_0' = \mathcal{D}(\tilde{\mathrm{T}}_{\mathrm{u}})$. Then

$$\begin{aligned}
\mathrm{U}' \tilde{\mathrm{T}}_u x &= \mathrm{U}' (\mathrm{T}_{ik} {\mathrm{M}'}^{jk} u_{\mathrm{tr}})_i^j \\
&= (\sum_j {\mathrm{U}'}^{lj} \mathrm{T}_{ik} {\mathrm{M}'}^{jk} u_{\mathrm{tr}})_i^l \\
&= (\sum_j \mathrm{T}_{ik} {\mathrm{U}'}^{lj} {\mathrm{M}'}^{jk} u_{\mathrm{tr}})_i^l \text{ (cf. [Bol, Prop.2.1.])} \\
&= (\mathrm{T}_{ik} \sum_j {\mathrm{U}'}^{lj} {\mathrm{M}'}^{jk} u_{\mathrm{tr}})_i^l \\
&= \tilde{\mathrm{T}}_u \mathrm{U}' x
\end{aligned}$$

This shows $\mathrm{U}' \tilde{\mathrm{T}}_u = \tilde{\mathrm{T}}_u \mathrm{U}'$ for every unitary $\mathrm{U}' \in \mathcal{R}'$, hence, since D_0' is a core for T_u,

$$\mathrm{U}' \mathrm{T}_u = \mathrm{T}_u \mathrm{U}' \quad \forall \mathrm{U}' \in \mathcal{U}(\mathcal{R}').$$

5. In the last step we calculate

$$\begin{aligned}
\sum_k \mathrm{T}_u v_k &= \sum_k (\mathrm{T}_{ik} \delta^{jk} u_{tr})_i^j \\
&= (\mathrm{T}_{ij} u_{tr})_i^j = u.
\end{aligned}$$

□

Now we can give the following definition:

Definition 2.1. For every vector $u = (u_i^j) \in \mathcal{K}$ we denote by $\mathrm{T}_u = (\mathrm{T}_{ij})$ an operator affiliated with $\mathcal{R}$ s.t. $u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{T_{ij}})$ for all $i, j \in \mathbb{N}$, $\mathrm{T}_{ij} u_{\mathrm{tr}} = u_i^j$, and $\sum_k \mathrm{T}_u v_k = u$, which exists according to Lemma 2.1.

The next proposition shows some usefull properties of the operators occuring in Definition 2.1

Proposition 2.2. *Let* $\mathrm{T} \eta \mathcal{R}$, $v_k \in \mathcal{D}(\mathrm{T})$ *(*$k \in \mathbb{N}$*),* $\sum_k \|\mathrm{T} v_k\|^2 < \infty$*. Then*

1. $\mathcal{R} v_k \subset \mathcal{D}(\mathrm{T})$*,* $\mathcal{R} v_k \subset \mathcal{D}(\mathrm{T}^*)$*, and* $\mathcal{R} v_k \subset \mathcal{D}((\mathrm{T}^*\mathrm{T})^{1/2})$ *for all* $k \in \mathbb{N}$*.*
2. D_0 *is a core for* T, T^**, and* $(\mathrm{T}^*\mathrm{T})^{1/2}$.
3. $\mathcal{R}' v_k \subset \mathcal{D}(\mathrm{T})$*,* $\mathcal{R}' v_k \subset \mathcal{D}(\mathrm{T}^*)$*, and* $\mathcal{R}' v_k \subset \mathcal{D}((\mathrm{T}^*\mathrm{T})^{1/2})$ *for all* $k \in \mathbb{N}$*.*
4. D_0' *is a core for* T, T^**, and* $(\mathrm{T}^*\mathrm{T})^{1/2}$.

Proof. 1. Let $\mathrm{T} = \mathrm{VH}$ the polar decomposition of T, and E_λ the spectral resolution of H. Then

$$\begin{aligned}
\int \lambda^2 d \left\|\mathrm{E}_\lambda \mathrm{U} v_k\right\|^2 &\leq \sum_l \int \lambda^2 d \left\|\mathrm{E}_\lambda \mathrm{U} v_l\right\|^2 \\
&= \sum_l \int \lambda^2 d \left\|\mathrm{U}^* \mathrm{E}_\lambda v_l\right\|^2 \text{ (s. (2.1))} \\
&= \sum_l \int \lambda^2 d \left\|\mathrm{E}_\lambda v_l\right\|^2 \\
&= \sum_l \left\|\mathrm{H} v_l\right\|^2 = \sum_l \left\|\mathrm{T} v_l\right\|^2 < \infty
\end{aligned}$$

for every unitary $\mathrm{U} \in \mathcal{R}$ and every $k \in \mathbb{N}$, i.e. $\mathcal{R} v_k \subset \mathcal{D}(\mathrm{H}) = \mathcal{D}(\mathrm{T})$ for every $k \in \mathbb{N}$. Now $\mathrm{T}^* = \mathrm{HV}^*$, and, since $\mathrm{V}^* \in \mathcal{R}$, also $\mathcal{R} v_k \subset \mathcal{D}(\mathrm{T}^*)$.

2. 1) shows that $D_0 \subset \mathcal{D}(\mathrm{T})$, further D_0 is dense in $\mathcal{K}$. Now D_0 is invariant under the unitary group $e^{it\mathrm{H}}$, i.e. D_0 is a core for H and also for T. The assertion for T^* follows analogous.

3. This follows from 1) and [Bol, Prop. 2.1.].

4. Now for every $\mathrm{M} = (\mathrm{M}_{ij})_{ij} \in \mathcal{R}$ there exists exactly one $\mathrm{M}' = (\mathrm{M}'^{ij}) \in \mathcal{R}'$ s.t. $\mathrm{M}'^{ij} u_{\mathrm{tr}} = \mathrm{M}_{ji} u_{\mathrm{tr}}$ ($\mathrm{M}'^{ij} := \mathrm{J} \mathrm{M}_{ji} \mathrm{J}$, where J is the conjugation w.r.t. u_{tr}). Now define

$$\mathrm{M}'_{(k,l)} := \mathrm{E}'_k \mathrm{M}' \mathrm{E}'_l,$$

where $\mathrm{E}'_{(k)} := (\delta_{ik} \delta^{ij})^{ij}$. Then

$$\mathrm{M} v_k = \sum_l \mathrm{M}'_{(k,l)} v_l \tag{2.4}$$

and

$$\begin{aligned}
\sum_l \left\|\mathrm{T} \mathrm{M}'_{(k,l)} v_l\right\|^2 &= \sum_l \left\|\mathrm{M}'_{(k,l)} \mathrm{T} v_l\right\|^2 \\
&\leq \sum_l \left\|\mathrm{M}'\right\|^2 \left\|\mathrm{T} v_l\right\|^2 \\
&\leq \left\|\mathrm{M}\right\|^2 \sum_l \left\|\mathrm{T} v_l\right\|^2 < \infty.
\end{aligned}$$

Hence $\sum_l \mathrm{T} \mathrm{M}'_{(k,l)} v_l$ converges and therefore $\sum_l \mathrm{M}'_{(k,l)} v_l$ converges in the graph norm of T to $\mathrm{M} v_k$, i.e. also D'_0 is a core, since D_0 is it.

□

Lemma 2.3. *Let* $\mathrm{T} \eta \mathcal{R}$ *be as in Proposition 2.2. Then there are* $\mathrm{T}_{ij} \eta \mathcal{T}$ *with* $u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{T_{ij}})$ *s.t.* $\mathrm{T} = (\mathrm{T}_{ij})_{i,j \in \mathbb{N}}$ *and* $\mathrm{T} = \mathrm{T}_u$ *with* $u := \sum_k \mathrm{T} v_k$ *in the sense of Definition 2.1.*

Proof. Set $\mathrm{E}_k := [\mathcal{R}' v_k] \in \mathcal{R}$ ($k \in \mathbb{N}$). Then matrix calculation shows that (E_k) is a family of orthogonal, equivalent, finite projections, s.t.

$$\mathrm{E}_k \mathcal{R} \mathrm{E}_k = (\delta_{ki}\delta_{ij}\mathcal{T})_{i,j\in\mathbb{N}}$$

and $\sum_k \mathrm{E}_k = \mathrm{Id}$. Now $\mathrm{E}_{ij} : \mathrm{M}' v_j \mapsto \mathrm{M}' v_i$ defines a selfadjoint system of matrix units (E_{ij}) s.t. $\mathrm{E}_{kk} = \mathrm{E}_k$ for every $k \in \mathbb{N}$. Now define operators

$$\begin{aligned} \mathrm{S}_{ij} : \mathcal{D}(\mathrm{S_{ij}}) := D_0' \subset \mathcal{K} \to \mathcal{K} \\ \sum_k \mathrm{M}_k' v_k \mapsto \sum_k \mathrm{E}_{ki}\mathrm{T}\mathrm{E}_{jk}\mathrm{M}_k' v_k \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathrm{S}}_{ji} : \mathcal{D}(\tilde{\mathrm{S}}_{\mathrm{ij}}) := D_0' \subset \mathcal{K} \to \mathcal{K} \\ \sum_k \mathrm{M}_k' v_k \mapsto \sum_k \mathrm{E}_{kj}\mathrm{T}^*\mathrm{E}_{ik}\mathrm{M}_k' v_k. \end{aligned}$$

Since D_0' is dense in $\mathcal{K}$ (cf. proof of Lemma 2.1) and a core both for T and for T^* they are well defined and densely defined. Let now $x \in \mathcal{D}(\mathrm{S_{ij}})$ and $y \in \mathcal{D}(\tilde{\mathrm{S}}_{\mathrm{ji}})$. Then

$$\begin{aligned} \langle \mathrm{S}_{ij}x|y\rangle &= \sum_k \langle \mathrm{E}_{ki}\mathrm{T}\mathrm{E}_{jk}x|y\rangle \\ &= \sum_k \langle x|\mathrm{E}_{kj}\mathrm{T}^*\mathrm{E}_{ik}y\rangle \\ &= \langle x|\tilde{\mathrm{S}}_{ji}y\rangle. \end{aligned}$$

This means that $y \in \mathcal{D}(\mathrm{S_{ij}^*})$ and $\mathrm{S}_{ij}^* y = \tilde{\mathrm{S}}_{ji} y$, i.e. $\tilde{\mathrm{S}}_{ji} \subset \mathrm{S}_{ij}^*$, hence S_{ij} is closable since $\tilde{\mathrm{S}}_{ji}$ is densely defined.

Let now $\tilde{\mathrm{T}}_{ij}$ be the closure of S_{ij}. Then $D_0' = \mathcal{D}(\mathrm{S_{ij}})$ is a core for $\tilde{\mathrm{T}}_{ij}$. Since $\mathrm{U}' D_0' = D_0'$ and

$$\begin{aligned} \mathrm{U}'\mathrm{S}_{ij}(\sum_k \mathrm{M}_k' v_k) &= \sum_k \mathrm{U}'\mathrm{E}_{ki}\mathrm{T}\mathrm{E}_{jk}\mathrm{M}_k' v_k \\ &= \sum_k \mathrm{E}_{ki}\mathrm{T}\mathrm{E}_{jk}\mathrm{U}'\mathrm{M}_k' v_k \\ &= \mathrm{S}_{ij}\mathrm{U}'(\sum_k \mathrm{M}_k' v_k) \end{aligned}$$

for every unitary $\mathrm{U}' \in \mathcal{R}'$ and every element $(\mathrm{M}_k')v_k \in D_0'$, it follows that $\mathrm{U}'\tilde{\mathrm{T}}_{ij} = \tilde{\mathrm{T}}_{ij}\mathrm{U}'$ and $\tilde{\mathrm{T}}_{ij}$ is affiliated with $\mathcal{R}$.

Further

$$\begin{aligned} \mathrm{E}_{mn}\mathrm{S}_{ij}(\sum_k \mathrm{M}_k' v_k) &= \sum_k \mathrm{E}_{mn}\mathrm{E}_{ki}\mathrm{T}\mathrm{E}_{jk}\mathrm{M}_k' v_k \\ &= \mathrm{E}_{mi}\mathrm{T}\mathrm{E}_{jm}\mathrm{E}_{mn}\mathrm{M}_n' v_n \\ &= \sum_k \mathrm{E}_{ki}\mathrm{T}\mathrm{E}_{jk}\mathrm{E}_{mn}\mathrm{M}_n' v_n \\ &= \mathrm{S}_{ij}\mathrm{E}_{mn}(\sum_k \mathrm{M}_k' v_k), \end{aligned}$$

hence $\tilde{\mathrm{T}}_{ij}$ is affiliated with $\mathcal{T} \otimes \mathbb{C} \otimes \mathbb{C} = \{\mathrm{E}_{mn} | m, n \in \mathbb{N}\}' \cap \mathcal{R}$. Now set $\mathrm{T}_{ij} := \mathrm{V}^* \tilde{\mathrm{T}}_{ij} \mathrm{V}$, where

$$\begin{aligned} \mathrm{V} : \mathcal{H} &\to \mathcal{K} \\ v &\mapsto (\delta_{1i} \delta_i^j v)_i^j \end{aligned}$$

is the canonical partial isometry from $\mathcal{H}$ to $\mathcal{K} = \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty$.

Now $u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{T_{ij}})$ since $\mathrm{V} u_{\mathrm{tr}} = v_1 \in \mathcal{D}(\mathrm{T_{ij}})$, and with $u_i^j := \mathrm{T}_{ij} u_{\mathrm{tr}}$

$$\begin{aligned} \sum_{i,j} \left\| u_i^j \right\|^2 &= \sum_{i,j} \| \mathrm{T}_{ij} u_{\mathrm{tr}} \|^2 \\ &= \sum_{i,j} \left\| \mathrm{V}^* \tilde{\mathrm{T}}_{ij} \mathrm{V} u_{\mathrm{tr}} \right\|^2 \\ &= \sum_{i,j} \| \mathrm{E}_{1i} \mathrm{T} \mathrm{E}_{j1} v_1 \|^2 \\ &= \sum_{i,j} \| \mathrm{E}_i \mathrm{T} v_j \|^2 \\ &= \sum_j \| \mathrm{T} v_j \|^2 < \infty \end{aligned}$$

s.t. $u := \sum_k \mathrm{T} v_k = (u_i^j)_i^j = (\mathrm{T}_{ij} u_{\mathrm{tr}})_i^j \in \mathcal{K}$. This means that we can construct the operator $\mathrm{T}_u = (\mathrm{T}_{ij})_{ij}$ according to Lemma 2.1. Now T_u and T coincide on the core D_0', and hence they are equal. □

Corollary 2.4. *The operator* T_u *defined in Definition 2.1 is unique.*

Corollary 2.5. *Let* T_u *be the operator defined in Definition 2.1. Then* $\mathcal{R} v_k \in \mathcal{D}(\mathrm{T_u})$ *for every* $k \in \mathbb{N}$ *and*

$$\mathrm{T}_u \mathrm{M} v_k = (\sum_l \mathrm{T}_{il} \mathrm{M}_{lk} \delta_k^j u_{\mathrm{tr}})_i^j.$$

Proof. Proposition 2.2 shows that $\mathcal{R} v_k \in \mathcal{D}(\mathrm{T_u})$ for every $k \in \mathbb{N}$ and $\mathrm{M} v_k = \sum_l \mathrm{M}'_{(k,l)} v_l$ (cf. (2.4)). Now

$$\begin{aligned} \mathrm{T}_u \mathrm{M} v_k &= \mathrm{T}_u \sum_l \mathrm{M}'_{(k,l)} v_l \\ &= \sum_l \mathrm{T}_u \mathrm{M}'_{(k,l)} v_l \\ &= \sum_l (\mathrm{T}_{il} \mathrm{M}'^{k,l} \delta_{jk} u_{\mathrm{tr}})_i^j \\ &= \sum_l (\mathrm{T}_{il} \mathrm{M}_{lk} \delta_k^j u_{\mathrm{tr}})_i^j. \end{aligned}$$

□

Now we can formulate the following lemma:

Lemma 2.6. *Let* T_u *be the operator defined in Definition 2.1. Then:*

1. $\operatorname{tr}(\mathrm{T}_u^*\mathrm{T}_u) = \operatorname{tr}(\mathrm{T}_u\mathrm{T}_u^*) < \infty$.
2. *u is cyclic, iff T_u is injective.*
3. *u is separating, iff T_u has dense range.*
4. *u is cyclic and separating iff T_u is injective and has dense range, i.e. iff T_u is invertible.*

For the proof we need:

Proposition 2.7. *Let $\mathcal{T}$ be a (finite) von Neumann algebra with cyclic trace vector u_{tr}. Let further $\mathrm{S}, \mathrm{T}\eta\mathcal{T}$ with $u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{S}) \cap \mathcal{D}(\mathrm{T})$ and $\mathrm{M}, \mathrm{N} \in \mathcal{T}$. Then*

$$\langle \mathrm{MT}u_{\mathrm{tr}} | \mathrm{NS}u_{\mathrm{tr}} \rangle = \langle \mathrm{S}^*\mathrm{N}^*u_{\mathrm{tr}} | \mathrm{T}^*\mathrm{M}^*u_{\mathrm{tr}} \rangle. \tag{2.5}$$

Proof. Let (E_n) and (F_n) be bounding sequences for T and S, resp. (cf. [KR83, Lem. 5.6.14]). Then:

$$\begin{aligned}
\langle \mathrm{MT}u_{\mathrm{tr}} | \mathrm{NS}u_{\mathrm{tr}} \rangle &= \lim_{n\to\infty} \langle \mathrm{MTE}_n u_{\mathrm{tr}} | \mathrm{NSF}_n u_{\mathrm{tr}} \rangle \\
&= \lim_{n\to\infty} \langle (\mathrm{SF}_n)^*\mathrm{N}^*u_{\mathrm{tr}} | (\mathrm{TE}_n)^*\mathrm{M}^*u_{\mathrm{tr}} \rangle \\
&= \lim_{n\to\infty} \langle \mathrm{F}_n\mathrm{S}^*\mathrm{N}^*u_{\mathrm{tr}} | \mathrm{E}_n\mathrm{T}^*\mathrm{M}^*u_{\mathrm{tr}} \rangle \\
&= \langle \mathrm{S}^*\mathrm{N}^*u_{\mathrm{tr}} | \mathrm{T}^*\mathrm{M}^*u_{\mathrm{tr}} \rangle,
\end{aligned}$$

since $\mathrm{N}^*u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{S}^*)$ and $\mathrm{M}^*u_{\mathrm{tr}} \in \mathcal{D}(\mathrm{T}^*)$ (cf. [Bol, Prop2.1]). ☐

Proof of Lemma 2.6. 1. Since $v_k \in \mathcal{D}(\mathrm{T_u}) = \mathcal{D}(\mathrm{H})$ for all $k \in \mathbb{N}$, where $\mathrm{T}_u = \mathrm{VH}$ is the polar decomposition of T_u, we can write the trace, defined in (2.1), as follows (E_λ is the spectral measure of H):

$$\operatorname{tr}(\mathrm{T}_u^*\mathrm{T}_u) = \operatorname{tr}(\mathrm{H}^2) = \sum_k \int \lambda^2 d \left\|\mathrm{E}_\lambda v_k\right\|^2 = \sum_k \left\|\mathrm{H}v_k\right\|^2 = \sum_k \left\|\mathrm{T}_u v_k\right\|^2.$$

Since the $[\mathcal{R}v_k]$ are mutually orthogonal, we have

$$\begin{aligned}
\operatorname{tr}(\mathrm{T}_u^*\mathrm{T}_u) &= \sum_k \langle \mathrm{T}_u v_k | \mathrm{T}_u v_k \rangle \\
&= \sum_{k,j} \langle \mathrm{T}_u v_k | \mathrm{T}_u v_j \rangle \\
&= \left\| \sum_k \mathrm{T}_u v_k \right\|^2 = \|u\|^2 < \infty.
\end{aligned}$$

Further

$$\begin{aligned}
\operatorname{tr}(\mathrm{T}_u\mathrm{T}_u^*) &= \sum_j \left\langle \mathrm{T}_u^* v_j | \mathrm{T}_u^* v_j \right\rangle \\
&= \sum_j \sum_{k,i} \left\langle \mathrm{T}_{jk}^* \delta^{ji} u_{\mathrm{tr}} | \mathrm{T}_{jk}^* \delta^{ji} u_{\mathrm{tr}} \right\rangle \\
&= \sum_{k,i} \left\langle \mathrm{T}_{ik}^* u_{\mathrm{tr}} | \mathrm{T}_{ik}^* u_{\mathrm{tr}} \right\rangle \\
&= \sum_k \sum_i \left\langle \mathrm{T}_{ik} u_{\mathrm{tr}} | \mathrm{T}_{ik} u_{\mathrm{tr}} \right\rangle \\
&= \sum_k \left\langle \mathrm{T}_u v_k | \mathrm{T}_u v_k \right\rangle \\
&= \operatorname{tr}(\mathrm{T}_u^*\mathrm{T}_u).
\end{aligned}$$

2. Let u be cyclic. Then there are $\mathrm{M}^{(n)} = (\mathrm{M}_{ik}^{(n)}) \in \mathcal{R}$ with

$$\lim_{n\to\infty} \mathrm{M}^{(n)} u = v$$

for every $v = (\mathrm{S}_{ij} u_{\mathrm{tr}})_i^j \in \mathcal{K}$, where $\mathrm{S} = (\mathrm{S}_{ij}) \in \mathcal{R}_0$. This means, using Proposition 2.7 and Corollary 2.5,

$$\begin{aligned}
0 \xleftarrow{\infty\leftarrow n} & \sum_{i,j} \left\| \sum_k \mathrm{M}_{ik}^{(n)} \mathrm{T}_{kj} u_{\mathrm{tr}} - \mathrm{S}_{ij} u_{\mathrm{tr}} \right\|^2 \\
&= \sum_{i,j} \left(\left\| \sum_k \mathrm{M}_{ik}^{(n)} \mathrm{T}_{kj} u_{\mathrm{tr}} \right\|^2 - 2 \sum_k \operatorname{Re}\left\langle \mathrm{M}_{ik}^{(n)} \mathrm{T}_{kj} u_{\mathrm{tr}} | \mathrm{S}_{ij} u_{\mathrm{tr}} \right\rangle + \left\| \mathrm{S}_{ij} u_{\mathrm{tr}} \right\|^2 \right) \\
&= \sum_{i,j} \left(\left\| \sum_k \mathrm{T}_{kj}^* (\mathrm{M}_{ik}^{(n)})^* u_{\mathrm{tr}} \right\|^2 - 2 \sum_k \operatorname{Re}\left\langle \mathrm{T}_{kj}^* (\mathrm{M}_{ik}^{(n)})^* u_{\mathrm{tr}} | \mathrm{S}_{ij}^* u_{\mathrm{tr}} \right\rangle + \left\| \mathrm{S}_{ij}^* u_{\mathrm{tr}} \right\|^2 \right) \\
&= \sum_{i,j} \left\| \sum_k \mathrm{T}_{kj}^* (\mathrm{M}_{ik}^{(n)})^* u_{\mathrm{tr}} - \mathrm{S}_{ij}^* u_{\mathrm{tr}} \right\|^2,
\end{aligned}$$

i.e., since $(\mathrm{T}_{ki}^*)_{i,k} \subset \mathrm{T}_u^*$ and $\mathcal{R}_0 u_{\mathrm{tr}}$ is dense in $\mathcal{K}$, T_u^* has dense range, i.e. T_u is injective.

Let now T_u be injective and $\mathrm{M}' = (\mathrm{M}'^{ij}) \in \mathcal{R}'$ with $\mathrm{M}'u = 0$. Now

$$\mathrm{M}'u = (\sum_j \mathrm{M}'^{ij} \mathrm{T}_{kj} u_{\mathrm{tr}})_{ik} = 0,$$

and

$$\begin{aligned}
0 = \left\| \mathrm{M}'u \right\|^2 &= \sum_{i,k} \left\| \sum_j \mathrm{M}'^{ij} \mathrm{T}_{kj} u_{\mathrm{tr}} \right\| \\
&= \sum_{i,k} \left\| \sum_j \mathrm{T}_{kj} \mathrm{M}'^{ij} u_{\mathrm{tr}} \right\| = \left\| \mathrm{T}_u v \right\|,
\end{aligned}$$

where $v := ({\mathrm{M}'}^{ij} u_{\mathrm{tr}})_i^j = \sum_k \mathrm{M}' v_k \in \mathcal{D}(\mathrm{T_u})$ (T_u is closed), hence $\mathrm{T}_u v = 0$, and, since T_u is injective, $v_i^j = {\mathrm{M}'}^{ij} u_{\mathrm{tr}} = 0$ for all $i, j \in \mathbb{N}$. Because u_{tr} is cyclic for $\mathcal{T}$ hence separating for $\mathcal{T}'$, ${\mathrm{M}'}^{ij} = 0$ for all $i, j \in \mathbb{N}$, s.t. $\mathrm{M}' = 0$.

3. Let u be separating. This means that u is cyclic for $\mathcal{R}'$. Then there are $\mathrm{M}_{(n)} = (\mathrm{M}_{(n)}^{ik}) \in \mathcal{R}'$ and

$$\lim_{n\to\infty} \mathrm{M}_{(n)} u = v$$

for every $v = (\mathrm{S}_{ij} u_{\mathrm{tr}})_i^j \in \mathcal{K}$, where $(\mathrm{S}_{ij}) \in \mathcal{R}_0'$ $(i, j \in \mathbb{N})$. This means

$$\begin{aligned} 0 \xleftarrow{\infty \leftarrow n} & \sum_{i,j} \left\| \sum_k \mathrm{M}_{(n)}^{jk} \mathrm{T}_{ik} u_{\mathrm{tr}} - \mathrm{S}_{ij} u_{\mathrm{tr}} \right\|^2 \\ &= \sum_{i,j} \left\| \sum_k \mathrm{T}_{ik} \mathrm{M}_{(n)}^{jk} u_{\mathrm{tr}} - \mathrm{S}_{ij} u_{\mathrm{tr}} \right\|^2 . \end{aligned}$$

Since $\mathcal{R}_0' u_{\mathrm{tr}}$ is dense in $\mathcal{K}$ we have proven that T_u has dense range.

For the converse read the argument backwards.

4. This follows from 2. and 3.

□

Remark 2.1. Also here, as in the finite case, the finite trace condition of Lemma 2.6 is not only necessary but also sufficient for an operator being the operator associated with a vector in the sense of Definition 2.1. Suppose that $\mathrm{tr}(\mathrm{T}^*\mathrm{T}) < \infty$ with $\mathrm{T}\eta\mathcal{R}$. Then

$$\begin{aligned} \infty > \mathrm{tr}(\mathrm{T}^*\mathrm{T}) &= \mathrm{tr}(\mathrm{H}^2) \\ &= \sum_k \int \lambda^2 d \left\| \mathrm{E}_\lambda v_k \right\| \end{aligned}$$

hence

$$\int \lambda^2 d \left\| \mathrm{E}_\lambda v_k \right\| < \infty \quad \forall k \in \mathbb{N},$$

i.e. $v_k \in \mathcal{D}(\mathrm{H}) = \mathcal{D}(\mathrm{T})$, and

$$\sum_k \left\| \mathrm{T} v_k \right\|^2 = \sum_k \left\| \mathrm{H} v_k \right\|^2 = \sum_k \int \lambda^2 d \left\| \mathrm{E}_\lambda v_k \right\| < \infty.$$

This shows that the assumptions of Lemma 2.3 are fulfilled.

Corollary 2.8. *$\mathcal{R}$ possesses a cyclic and separating vector $u_0 \in \mathcal{K}$.*

Proof. Set $\mathrm{T} := (\delta_{ij} j^{-2} \mathrm{Id})_{i,j}$ or $u_0 := \sum_j j^{-2} v_j$. Then T fulfills the conditions of Lemma 2.3 and is invertible, s.t. from Lemma 2.6 follows that u_0 is cyclic and separating. □

In the last step of this subsection we show that the model we have just treated is really representative for the general situation, in the sense that all infinite type I or type II factors can be considered as a matrix algebra of finite type I or type II factors, resp. This is shown by the next

Lemma 2.9. *Every infinite but semifinite von Neumann factor $(\mathcal{M}_0, \mathcal{H}_0)$ with cyclic and separating vector $u_0 \in \mathcal{H}_0$ is unitarily equivalent to $\mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C} =: \mathcal{R}, \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty =: \mathcal{K})$, where $\mathcal{T}$ is a finite von Neumann factor acting on the Hilbert space $\mathcal{H}$ with cyclic and separating vector and $\mathcal{H}_\infty$ is a separable infinite dimensional Hilbert space.*

Proof. Since $\mathcal{M}_0$ is infinite but semifinite there is a countable orthogonal family of finite equivalent projections $(\mathrm{E}_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_0$, s.t. $\sum \mathrm{E}_n = \mathrm{Id}$. Now there is a selfadjoint system of matrix units $(\mathrm{E}_{ab})_{a,b\in\mathbb{N}}$ with $\mathrm{E}_{aa} = \mathrm{E}_a$ (cf. [KR86, 6.6.4]). This shows that $\mathcal{M}_0$ is isomorphic to $\tilde{\mathcal{T}} \otimes L(\mathcal{H}_\infty)$ where $\tilde{\mathcal{T}} := \{\mathrm{E}_{ab}\}' \cap \mathcal{M}_0$ and $\tilde{\mathcal{T}}$ is isomorphic to every $\mathrm{E}_n\mathcal{M}_0\mathrm{E}_n$ $(n \in \mathbb{N})$. Since the projections E_n are finite also $\tilde{\mathcal{T}}$ is a finite factor.

Since $\mathcal{M}_0$ possesses the separating vector u_0 we can represent the algebras $\mathrm{E}_n\mathcal{M}_0\mathrm{E}_n$ by the GNS representation for the faithful state ω_n induced by the separating vector $\mathrm{E}_n u_0$ on a Hilbert space $\mathcal{H}_n$, s.t. the vector $u_n \in \mathcal{H}_n$ implementing the state ω_n is a cyclic and separating vector for $\mathrm{E}_n\mathcal{M}_0\mathrm{E}_n$. Since all the $\mathrm{E}_n\mathcal{M}_0\mathrm{E}_n$ are isomorphic and they possess in this representation a cyclic and separating vector, they are all unitarily equivalent. This means that we can choose as $\mathcal{T}$ one of the $\mathrm{E}_n\mathcal{M}_0\mathrm{E}_n$ acting on the representation space $\mathcal{H}_n$.

Since the factor $(\mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C} =: \mathcal{R}, \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty =: \mathcal{K})$ possesses a cyclic and separating vector if $(\mathcal{T}, \mathcal{H})$ does (see Corollary 2.8) and it is isomorphic to $\mathcal{M}_0$ it is unitarily equivalent to $\mathcal{M}_0$. □

The results of this section (and the analogues in [Bol]) can be subsumed in the next two theorems:

Theorem 2.10. *Let $(\mathcal{M}_0, \mathcal{H}_0)$ be a semifinite von Neumann factor. Let further $u \in \mathcal{H}_0$. Then there is exactly one operator $\mathrm{T}_u \eta \mathcal{M}_0$ associated with the vector u in the sense of [Bol, Def 2.1.] in the finite case and in the sense of Definition 2.1 in the infinite case, resp., having the following properties:*

1. $\mathrm{tr}(\mathrm{T}_u\mathrm{T}_u^*) = \mathrm{tr}(\mathrm{T}_u^*\mathrm{T}_u) < \infty$.
2. *u is cyclic, iff T_u is injective.*
3. *u is separating, iff T_u has dense range.*
4. *u is cyclic and separating iff T_u is injective and has dense range, i.e. iff T_u is invertible.*

Proof. The finite case is just Theorem 1.1. In the infinite case the existence and the asserted properties follow from Lemma 2.9 and Lemma 2.6 infinite case, the uniqueness from Corollary 2.4. □

Theorem 2.11. *Let* $\mathrm{T}\eta\mathcal{M}_0$*. Then there is a vector* $u \in \mathcal{H}_0$ *s.t* $\mathrm{T} = \mathrm{T}_u$ *iff* $\mathrm{tr}(\mathrm{T}\mathrm{T}^*) = \mathrm{tr}(\mathrm{T}^*\mathrm{T}) < \infty$*.*

Proof of Theorem 2.11. Again the finite case is just Theorem 1.2. In the infinite case the necessarity of the trace condition follows from Theorem 2.10 and the sufficiency from Remark 2.1, resp. □

3 Generation of Modular Objects

In this section we show how the modular objects of a cyclic and separating vector $u_0 \in \mathcal{H}$ for a semifinite von Neumann factor $(\mathcal{M}_0, \mathcal{H}_0)$ are related to the operator T_{u_0} constructed in the last section. As in §2 we consider as a model for the infinite but semifinite factor the factor $\mathcal{T} \otimes L(\mathcal{H}_\infty) \otimes \mathbb{C} =: \mathcal{R}, \mathcal{H} \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty) =: \mathcal{K})$, where $\mathcal{T}$ is a finite factor with cyclic trace vector $u_{\mathrm{tr}} \in \mathcal{H}$. If $u_0 \in \mathcal{K}$ is a cyclic and separating vector for $\mathcal{R}$, according to Lemma 2.6, there is an invertible operator $\mathrm{T}_{u_0}\eta\mathcal{R}$, s.t. $u_0 = \sum_k \mathrm{T}_{u_0} v_k$, where $v_k = (\delta_i^j \delta_{ik} u_{\mathrm{tr}})_i^j$. Using this operator we can formulate the following analogue to Theorem 1.3:

Theorem 3.1. *Use the notations from above. Let further*

$$\mathrm{T}_{u_0} = \mathrm{HV} = (\mathrm{H_{ij}})_{ij} (\mathrm{V}_{ij})_{ij}$$

be the polar decomposition of T_{u_0}*. With the conjugation* $\tilde{\mathrm{J}}$ *defined as*

$$\tilde{\mathrm{J}}(\mathrm{M}_{ij} u_{\mathrm{tr}})_i^j := (\mathrm{M}_{ji}^* u_{\mathrm{tr}})_i^j := (\mathrm{J}\mathrm{M}_{ji} u_{\mathrm{tr}})_i^j \quad \forall \mathrm{M} = (\mathrm{M}_{ij})_{ij} \in \mathcal{R},$$

where J *is the conjugation corresponding to the trace vector* u_{tr}*, we can calculate the modular objects* (Δ_0, J_0) *of* $(\mathcal{M}_0, u_0)$ *as follows:*

$$\mathrm{J}_0 = \tilde{\mathrm{J}}\mathrm{V}^*\tilde{\mathrm{J}}\mathrm{V}\tilde{\mathrm{J}} = \mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*,$$

and

$$\Delta_0 = \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \mathrm{H}_0,$$

where $\mathrm{H}_0 = \mathrm{H}^2 = \mathrm{T}_{u_0}\mathrm{T}_{u_0}^*$*.*

Proof. 1. First we observe that $\tilde{\mathrm{J}}\mathrm{R}\tilde{\mathrm{J}} \in \mathcal{R}'$ for every $\mathrm{R} \in \mathcal{R}$. For let $\mathrm{R} = (\mathrm{R}_{ij}) \in \mathcal{R}$ and $v = (u_i^j)_i^j = (\mathrm{M}_{ij} u_{\mathrm{tr}})_i^j \in \mathcal{K}$, $(\mathrm{M}_{ij}) \in \mathcal{R}_0$, then

$$\begin{aligned}
\tilde{\mathrm{J}}\mathrm{R}\tilde{\mathrm{J}}v &= \tilde{\mathrm{J}}\mathrm{R}(\mathrm{J}\mathrm{M}_{ji} u_{\mathrm{tr}})_i^j \\
&= \tilde{\mathrm{J}}(\sum_i \mathrm{R}_{ki}\mathrm{J}\mathrm{M}_{ji} u_{\mathrm{tr}})_k^j \\
&= (\mathrm{J}\sum_i \mathrm{R}_{ji}\mathrm{J}\mathrm{M}_{ki} u_{\mathrm{tr}})_k^j \\
&= \underbrace{(\mathrm{J}\mathrm{R}_{ji}\mathrm{J})^{ji}}_{:=\mathrm{R}' \in \mathcal{R}'} (\mathrm{M}_{ki} u_{\mathrm{tr}})_k^i \\
&= \mathrm{R}' v.
\end{aligned}$$

Further

$$\tilde{\mathrm{J}}\tilde{\mathrm{J}}v = \tilde{\mathrm{J}}(\mathrm{J}\mathrm{M}_{ji}u_{\mathrm{tr}})_{ij} = (\mathrm{M}_{ij}u_{\mathrm{tr}})_{ij} = v,$$

s.t. $\tilde{\mathrm{J}}$ is an (algebraic) conjugation for $\mathcal{R}$.

2. Let now T_{u_0} be bounded ($\Rightarrow$ all the $\mathrm{T_{ij}}$ and H_{ij}, resp. are bounded). Then we show that the Tomita operator S defined by

$$\mathrm{SA}u_0 = \mathrm{A}^*u_0 \quad \forall \mathrm{A} \in \mathcal{R}$$

can be written as

$$\mathrm{S} = \mathrm{H}^{-1}\mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H}. \tag{3.1}$$

For this let $\mathrm{A} = (\mathrm{A}_{ij})_{ij} \in \mathcal{R}$ and $u_0 = (\sum_k \mathrm{H}_{jk}\mathrm{V}_{kl}u_{\mathrm{tr}})^l_j$. Then

$$\mathrm{A}u_0 = (\sum_{j,k} \mathrm{A}_{ij}\mathrm{H}_{jk}\mathrm{V}_{kl}u_{\mathrm{tr}})^l_i$$

and

$$\mathrm{A}^*u_0 = (\sum_{j,k} \mathrm{A}^*_{ji}\mathrm{H}_{jk}\mathrm{V}_{kl}u_{\mathrm{tr}})^l_i.$$

Now

$$\begin{aligned}
(\mathrm{H}^{-1}\mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H})\mathrm{A}u_0 &= \mathrm{H}^{-1}\mathrm{V}\tilde{\mathrm{J}}(\sum_{i,j,k,m} \mathrm{V}^*_{mn}\mathrm{H}_{mi}\mathrm{A}_{ij}\mathrm{H}_{jk}\mathrm{V}_{kl}u_{\mathrm{tr}})^l_n \\
&= \mathrm{H}^{-1}\mathrm{V}(\sum_{i,j,k,m} \mathrm{J}\mathrm{V}^*_{ml}\mathrm{H}_{mi}\mathrm{A}_{ij}\mathrm{H}_{jk}\mathrm{V}_{kn}u_{\mathrm{tr}})^l_n \\
&= \mathrm{H}^{-1}\mathrm{V}(\sum_{i,j,k,m} \mathrm{V}^*_{kn}\mathrm{H}^*_{jk}\mathrm{A}^*_{ij}\mathrm{H}^*_{mi}\mathrm{V}_{ml}u_{\mathrm{tr}})^l_n \\
&= \mathrm{H}^{-1}\mathrm{V}(\sum_{i,j,k,m} \mathrm{V}^*_{kn}\mathrm{H}_{kj}\mathrm{A}^*_{ij}\mathrm{H}_{im}\mathrm{V}_{ml}u_{\mathrm{tr}})^l_n \\
&= \mathrm{H}^{-1}(\sum_{i,j,m} \mathrm{H}_{nj}\mathrm{A}^*_{ij}\mathrm{H}_{im}\mathrm{V}_{ml}u_{\mathrm{tr}})^l_n \\
&= (\sum_{i,m} \mathrm{A}^*_{in}\mathrm{H}_{im}\mathrm{V}_{ml}u_{\mathrm{tr}})^l_n = \mathrm{A}^*u_0,
\end{aligned}$$

which proves (3.1). Now $\mathrm{S}^* = \mathrm{H}\mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H}^{-1}$ and

$$\begin{aligned}
\Delta_0 &= \mathrm{S}^*\mathrm{S} \\
&= \mathrm{H}\mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H}^{-1}\mathrm{H}^{-1}\mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H} \\
&= \mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H}^{-2}\mathrm{V}\tilde{\mathrm{J}}\mathrm{V}^*\mathrm{H}^2 \\
&= \mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0.
\end{aligned}$$

Further

$$\mathrm{J}_0\Delta_0^{1/2} = \mathrm{H}^{-1}\mathrm{J}_0\mathrm{H} = \mathrm{S},$$

and all the assertions are proven in the bounded case.

3. In the last step we approximate the (unbounded) operator T_{u_0} by bounded operators T_n in exactly the same way as in the proof of Theorem 3.1. in [Bol] and show the assertions like there also in the unbounded case.

□

4 The Second Simple Class of Solutions of the Inverse Problem

In this section we want to use the results of the last two sections to examine the second simple classes of solutions of the inverse problem introduced by Wollenberg in [Wolb] for type I factors, and considered in [Bol] also for type II_1 factors. For the construction of this class it is crucial that the inverse Δ_0^{-1} of the modular operator is again a modular operator. To this scope there was shown the following

Lemma 4.1. *Let $\Delta_0 = \mathrm{J}_0\mathrm{H}_0^{-1}\mathrm{J}_0\mathrm{H}_0$ be the decomposition of the modular operator Δ_0, where $\mathrm{J}_0 = \mathrm{JV^*JVJ} = \mathrm{VJV^*}$ and $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}$ is the operator corresponding to u_0 (cf. Theorem 3.1). Then $\Delta_0^{-1} = \mathrm{J}_0\mathrm{H}_0\mathrm{J}_0\mathrm{H}_0^{-1}$ and the following is equivalent:*

1. *$(\Delta_0^{-1}, \mathrm{J}_0)$ are the modular objects w.r.t. a cyclic and separating vector $u_1 \in \mathcal{H}_0$.*
2. $$\mathrm{tr}(\mathrm{H}_0^{-1}) < \infty. \tag{4.1}$$

This lemma can be proven with the same techniques as in [Bol] also for the infinite case taking into account Theorem 2.10, Theorem 2.11, and Theorem 3.1.

Now we must examine, whether or not the second condition in Lemma 4.1 is fulfilled:

Lemma 4.2. *For type I_∞ and type II_∞ factors the condition* (4.1) *is never true.*

Proof. Let $\mathcal{M}_0$ now be a type I_∞ or II_∞ factor and $\mathrm{T}_{u_0} = \mathrm{H}_0^{1/2}\mathrm{V}$ the operator corresponding to the cyclic and separating vector u_0. Let further $\mathrm{E}_\lambda \in \mathcal{M}_0$ the spectral resolution of H_0. Then we can define a positive measure μ_{tr} on the σ-algebra of Borel sets in $\mathbb{R}$, s.t.

$$\mathrm{tr}(\mathrm{H}_0) = \int \lambda d\mu_{\mathrm{tr}}(\lambda),$$

where

$$\mu_{\mathrm{tr}}(B) := \mathrm{tr}\,\mathrm{E}(B)$$

for all Borel sets B. Now $c := \mathrm{tr}(\mathrm{H}_0) < \infty$. Assume w.l.o.g. $c = 1$. Then

$$1 = \int \lambda d\mu_{\mathrm{tr}}(\lambda) \geq \int_{[0,1]} \lambda d\mu_{\mathrm{tr}}(\lambda) + \int_{(1,\infty)} d\mu_{\mathrm{tr}}(\lambda),$$

i.e.

$$\int\limits_{(1,\infty)} d\mu_{\mathrm{tr}}(\lambda) < \infty.$$

Since $\mathcal{M}_0$ is infinite $\infty = \mathrm{tr}(\mathrm{Id}) = \mu_{\mathrm{tr}}(\mathbb{R})$, i.e.

$$\infty = \int\limits_{\lambda} d\mu_{\mathrm{tr}}(\lambda) = \int\limits_{[0,1]} d\mu_{\mathrm{tr}}(\lambda) + \underbrace{\int\limits_{(1,\infty)} d\mu_{\mathrm{tr}}(\lambda)}_{<\infty},$$

hence

$$\int\limits_{[0,1]} d\mu_{\mathrm{tr}}(\lambda) = \infty.$$

Suppose now that also $\mathrm{tr}(\mathrm{H}_0^{-1}) < \infty$, then

$$\infty > \int\limits_{\lambda} \lambda^{-1} d\mu_{\mathrm{tr}}(\lambda) \geq \underbrace{\int\limits_{[0,1]} d\mu_{\mathrm{tr}}(\lambda)}_{=\infty} + \int\limits_{(1,\infty)} \lambda^{-1} d\mu_{\mathrm{tr}}(\lambda),$$

which is a contradiction. □

Hence the last lemma shows that for infinite semifinite factors the second class of solutions of the inverse problem can never be constructed. This result was yet obtained by Wollenberg in [Wolb] for the type I_∞ case.

5 The Classification of Solutions in the Pure Point Spectrum Case

In this section we want to show the modifications of classification results obtained in [Bol]. The definition of the equivalence relation does not use any special properties of the finite factors, and can just be repeated here:

Definition 5.1. Two semifinite von Neumann factors $\mathcal{M}, \mathcal{N} \in NF_{\mathcal{M}_0}(\Delta_0, \mathrm{J}_0, u_0)$ are called equivalent, $\mathcal{M} \sim \mathcal{N}$, if $\mathcal{M} \in NF^1_{\mathcal{N}}(\Delta_0, \mathrm{J}_0, u_0)$, i.e. if there exists a unitary operator U on $\mathcal{H}_0$, s.t. $\mathcal{M} = \mathrm{U}\mathcal{N}\mathrm{U}^*$, U commutes with Δ_0 and J_0 and $\mathrm{U}^* u_0 = \pm u_0$ (For the definition of the class $NF^1_{\mathcal{N}}(\Delta_0, \mathrm{J}_0, u_0)$ see [Bol]).

Also the next lemmas can be formulated and proved in exactly the same way as in the finite case. Assume in the following that H_0 has pure point spectrum, i.e. $\mathrm{H}_0 = \sum_{k \in K} \mu_k \mathrm{E}_k$ where the μ_k $(k \in K)$ are the eigenvalues of H_0 and $\mathrm{E}_k \in \mathcal{M}_0$ are the corresponding (orthogonal) eigenprojections with

$m_k := \operatorname{tr} \mathrm{E}_k =: D_{\mathcal{M}_0}(\mathrm{E}_k)$ their von Neumann dimension. Then we have for Δ_0 the following decomposition

$$\begin{aligned} \Delta_0 &= \mathrm{H}_0 \mathrm{J}_0 \mathrm{H}_0^{-1} \mathrm{J}_0 \\ &= \sum_{k,l \in K} \mu_k \mu_l^{-1} \mathrm{E}_k \mathrm{J}_0 \mathrm{E}_l \mathrm{J}_0 \\ &= \sum_{j \in J} \lambda_j \mathrm{F}_j, \end{aligned} \tag{5.1}$$

where the λ_j $(j \in J)$ are the eigenvalues of Δ_0 and F_j are the corresponding eigenprojections. Now

Lemma 5.1. *With the notations introduced above we can compute the spectrum of Δ_0 in the following way:*

$$\{\lambda_j | j \in J\} = \{\mu_k \mu_l^{-1} | k, l \in K\} \quad \forall j \in J \tag{5.2}$$

and

$$n_j = \sum_{\mu_k \mu_l^{-1} = \lambda_j} m_k m_l \quad \forall j \in J \text{ if } \mathcal{M}_0 \text{ is type } I, \tag{5.3a}$$

$$n_j = \infty \quad \forall j \in J \text{ if } \mathcal{M}_0 \text{ is type } II, \tag{5.3b}$$

where $n_j := D_{L(\mathcal{H}_0)}(\mathrm{F}_j)$ with $D_{L(\mathcal{H}_0)}(\mathrm{F}_j)$ the dimension function in the type I_∞ factor $L(\mathcal{H}_0)$, which corresponds to the normalized Hilbert space dimension.

Lemma 5.2. *If there are two solutions of the inverse problem $\mathcal{M}_1$, $\mathcal{M}_2$ s.t. the corresponding selfadjoint operators H_1 and H_2 have the same eigenvalues modulo a positive constant $c > 0$ and same (von Neumann) multiplicities, then $\mathcal{M}_1 \sim \mathcal{M}_2$.*

Lemma 5.3. *If there are two equivalent solutions $\mathcal{M}_1$, $\mathcal{M}_2$ of the inverse problem with the corresponding positive operators H_1 and H_2, resp., (having pure point spectrum) then H_1 and H_2 have the same eigenvalues (up to a positive constant) and von Neumann multiplicities, i.e. they are unitarily equivalent in $\mathcal{M}_0$.*

The only difference to the finite case is shown by the next

Lemma 5.4. *Let $(\mu_k, m_k)_{k \in K}$ be a sequence of pairs of positive reals $\mu_k > 0$ and $m_k > 0$, s.t.*

$$m_k \in \mathbb{N} \text{ if } \mathcal{M}_0 \text{ is type } I_\infty, \tag{5.4a}$$

$$m_k \in \mathbb{R}_{>0} \text{ if } \mathcal{M}_0 \text{ is type } II_\infty, \tag{5.4b}$$

and

$$\sum_{k\in K} m_k = \infty \tag{5.4c}$$

and

$$\sum_{k\in K} m_k \mu_k = 1 \tag{5.4d}$$

and the relations (5.2) *and* (5.3) *are fulfilled. Then there exists a solution* $\mathcal{M} = \mathrm{U}\mathcal{M}_0\mathrm{U}^* \in NF_{\mathcal{M}_0}(\Delta_0, \mathrm{J}_0, u_0)$*, s.t.* $\mathrm{U}^*\Delta_0\mathrm{U} = \mathrm{H}\mathrm{J}_0\mathrm{H}^{-1}\mathrm{J}_0$ *and* H *has the eigenvalues and multiplicities* $(\mu_k, m_k)_{k\in K}$ *(cf. [Wolb, prop.4.1]).*

For the proof we need the following auxiliary results:

Proposition 5.5. *If* (m_k) *is countable family of positive reals with* $\sum m_k = \infty$*, then there exists in a type* II_∞ *von Neumann factor* $\mathcal{M}$ *a family of pairwise orthogonal projections* (E_k)*, s.t.* $D(\mathrm{E}_k) = m_k$ *for every* k*.*

Proof. We construct the E_k inductively: Since the range of $D_{\mathcal{M}}$ is all of $\mathbb{R}_{\leq 0}$ (cf. [KR86, 8.4.4]) there is a projection in $\mathcal{M}$, s.t. $D(\mathrm{E}_1) = m_1$.

Suppose now that for $N \in \mathbb{N}$ the E_k are pairwise orthogonal with $D_{\mathcal{M}_0}(\mathrm{E}_k) = m_k$ $(1 \leq k < N)$. Setting $\mathrm{F}_N := \mathrm{Id} - \sum_{k=1}^{N} \mathrm{E}_k$ the restricted algebra $\mathrm{F}_N\mathcal{M}\mathrm{F}_N$ is again a type II factor, finite, if F_N is finite, and infinite, if F_N is infinite (cf. [KR86, Ex. 6.9.16]) with the dimension function

$$D_N(\mathrm{F}_n\mathrm{E}\mathrm{F}_N) := D_{\mathcal{M}_0}(\mathrm{F}_n\mathrm{E}\mathrm{F}_N)/D(\mathrm{F}_N) \quad \forall \mathrm{F}_n\mathrm{E}\mathrm{F}_N \in \mathrm{F}_N\mathcal{M}\mathrm{F}_N,$$

if F_N is finite, and $D_N = D_{\mathcal{M}_0}$ else, where

$$D_{\mathcal{M}_0}(\mathrm{F}_N) = D_{\mathcal{M}_0}(\mathrm{Id} - \sum_{k=1}^{N} \mathrm{E}_k) = 1 - \sum_{k=1}^{N} D_{\mathcal{M}_0}(\mathrm{E}_k) \geq m_N.$$

With the same argument as above there is again a projection $\mathrm{E}_N \in \mathrm{F}_N\mathcal{M}\mathrm{F}_N \subset \mathcal{M}$, s.t. $D_N(\mathrm{E}_N) = D(\mathrm{F}_N)^{-1}m_N \leq 1$, if F_N is finite, and $D_N(\mathrm{E}_N) = m_N$ else. In both cases $D_{\mathcal{M}_0}(\mathrm{E}_N)$ and $\mathrm{E}_N < \mathrm{F}_N \perp \mathrm{E}_k$ $(1 \leq k < N)$. ∎

Now the proof of Lemma 5.4 is the same as in [Bol].

Remark 5.1. (5.4) show that in the infinite case we have always an infinite set of eigenvalues with 0 as cummulation point, i.e. $K = \mathbb{N}$ and 0 is in the spectrum $\sigma(\mathrm{H})$ of H.

Now we can summarize the lemmas of this section in the following

Theorem 5.6. *Let* $\mathcal{M}_0$ *be a semifinite von Neumann factor with cyclic and separating vector* u_0 *and* $\mathrm{T}_{u_0} = \mathrm{H}_0^{-1/2}\mathrm{V}$ *the operator corresponding to* u_0*. If* H_0 *has pure point spectrum, also* Δ_0 *has it. In this case let* (λ_j) $(j \in J)$ *be the eigenvalues of* Δ_0*. Then*

1. *Two solutions $\mathcal{M}_1, M_2 \in NF_{\mathcal{M}_0}(\Delta_0, \mathrm{J}_0, u_0)$ of the inverse problem with corresponding invertible operators $\mathrm{H}_i \eta \mathcal{M}_0$ $(i = 1, 2)$ having pure point spectrum are equivalent iff H_1 and H_2 have the same eigenvalues and (von Neumann) multiplicities.*

2. *A positive invertible operator $\mathrm{H}\eta\mathcal{M}_0$ with pure point spectrum gives rise to a solution of the inverse problem iff its eigenvalues and multiplicities satisfy* (5.2), (5.3), *and* (5.4).

3. *When the corresponding operators* H *has pure point spectrum the equivalence classes of $\sim$ are completely classified by the spectrum of the corresponding operators, i.e. by sequences of pairs of positive reals (μ_k, m_k) satisfying* (5.2), (5.3), *and* (5.4).

Example 5.1. Here we want to give some examples to illustrate Theorem 5.6.

1. In [Wolb] you can find some examples for the type I case.

2. Let

$$(\dots, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^3, \dots)$$

   be the eigenvalues of a modular operator for a type II_∞ factor. Then

$$((c_1 \cdot 1, 1), (c_1 \cdot 10^{-1}, 1), (c_1 \cdot 10^{-2}, 1), (c_1 \cdot 10^{-3}, 1), \dots)$$

   and

$$((c_2 \cdot 1, 1), (c_2 \cdot 10^{-1}, 1), (c_2 \cdot 10^{-3}, 1), (c_2 \cdot 10^{-5}, 1), \dots)$$

   characterize two different classes of solutions of the inverse problem, i.e. they both satisfy (5.2), (5.3), and (5.4), where c_i $(i = 1, 2)$ are appropriate chosen constants. This shows that in this case there are more than the simple classes of solutions of the inverse problem.

3. Let $(\mu_k, m_k)_{k\in\mathbb{N}}$ characterize a class of solutions of the inverse problem in the type II_∞ case, where $m_l \neq m_k$ for at least one pair $k, l \in \mathbb{N}$, then for every finite permutation σ of $\mathbb{N}$ interchanging k and l also $(c\mu_k, m_{\sigma(k)})$ characterize another class of solutions of the inverse problem ($c > 0$ a norming constant) which is really a new one.

4. Let again $(\mu_k, m_k)_{k\in\mathbb{N}}$ be a solution of the inverse problem in the type II_∞ case, and let $k, l \in \mathbb{N}$ be a pair of indices and $\epsilon > 0$. Then we get another class by adding ϵ to m_k and subtracting it from m_l where again we have really a new class.

Remark 5.2. 1. Example 5.1.3 and Example 5.1.4 shows that in the type II_∞ case, when H_0 has pure point spectrum, we can always construct a second class of solutions, different from the simple class discussed in §4, i.e. $NF_{\mathcal{M}_0} \neq NF^1_{\mathcal{M}_0}$, in contrast to the type I case, where for modular operators with generic spectrum we have $NF_{\mathcal{M}_0} = NF^1_{\mathcal{M}_0}$ (cf. [Wolb]).

2. Unfortunately the classification result presented here applies only to operators with pure point spectrum. Whereas in general there are also operators with more complicated spectrum (cf. [Bol, Remark 4.1]), for type I factors this is no restriction, since all operators generating modular operators are trace class operators, hence have pure point spectrum.

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